

Nonoscillation and Stability of the Second Order Ordinary Differential Equations with a Damping Term

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Abstract

In this paper we consider the linear ordinary equation of the second order

$$\mathcal{L}x(t) \equiv \ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(t) = f(t), \quad (0.1)$$

and the corresponding homogeneous equation

$$\ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(t) = 0. \quad (0.2)$$

Note that $[\alpha, \beta]$ is called a nonoscillation interval if every nontrivial solution has at most one zero on this interval. Many investigations which seem to have no connection such as differential inequalities, the Polia-Mammana decomposition (i.e. representation of the operator \mathcal{L} in the form of products of the first order differential operators), unique solvability of the interpolation problems, kernels oscillation, separation of zeros, zones of Lyapunov's stability and some others have a certain common basis - nonoscillation. Presumably Sturm was the first to consider the two problems which naturally appear here: to develop corollaries of nonoscillation and to find methods to check nonoscillation. In this paper we obtain several tests for nonoscillation on the semiaxis and apply them to propose new results on asymptotic properties and the exponential stability of the second order equation (0.2). Using the Floquet representations and upper and lower estimates of nonoscillation intervals of oscillatory solutions we deduce results on the exponential and Lyapunov's stability and instability of equation (0.2).

Keywords and Phrases: ordinary differential equations, nonoscillation interval, exponential stability, boundary value problems, Cauchy function, Green's function, Floquet theory.

AMS(MOS) subject classification: 34D, 34A30.

1 Introduction

This paper deals with the equation

$$\ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(t) = f(t), \quad t \in [0, \omega], \quad (1.1)$$

with locally summable coefficients a, b, f , which together with nonlinear equation

$$\ddot{x}(t) = g(t, x(t), \dot{x}(t)), \quad t \in [0, \omega], \quad (1.2)$$

continue to attract attention of many mathematicians due to their significance in applications. In this paper we obtain two group of results: on nonoscillation and on exponential stability. To obtain stability results we will use nonoscillatory equations as the so-called model equations for the left or right regularization and then use the classical Bohl-Perron theorem [11, 18]. Another approach to stability study for periodic equations in the present paper is based on lower and upper estimates of the distance between two adjacent zeros (i.e., of nonoscillation intervals) for nontrivial solutions of homogeneous equations. The foundations of this approach can be found in the work by Zhukovskii [40], Kreĭn [26] and Yakubovich [38]. Zones of Lyapunov's stability can be also studied on this basis. This explains why we connect the different areas together as well as the fact that actually our approach develops applications of the classical nonoscillation area. Note that we obtain new exponential stability conditions for equations with measurable coefficients. In most stability conditions it was assumed that $b(t) \equiv b > 0$ [19, 20, 28, 33, 35], $b(t) \geq 0$ is a differentiable function [8, 21, 23, 24, 31] or some restrictions like slow varying coefficients [13, 15, 16] were imposed. We consider here equation (1.1) without usual restrictions on parameters of the equations, the coefficients are even not required to be continuous.

Let us describe nonoscillation in general in order to understand how the results of the present paper develop also many other topics. The nonoscillation area consists of many topics which seem to have no relevance to each others but they are deeply connected.

The classical de la Vallée-Poussin theorem claims that existence of a positive function v such that $v''(t) + p(t)v(t) \leq 0$ for $t \in [0, \omega]$ implies that $[0, \omega]$ is a nonoscillation interval. The idea of theorems on differential inequalities can be formulated as follows: under certain conditions solutions of inequalities are greater or less than the solution of the equation. The idea to construct an approximate integration method for the numerical solution of differential equations based on the comparison of solutions of equations and inequalities first appeared in the works of famous Russian mathematician Chaplygin [9] and later was developed by other famous Russian mathematician Luzin [29]. Concerning our object we can formulate the differential inequality theorem in the form: *under certain conditions the inequalities*

$$(\mathcal{L}y)(t) \geq (\mathcal{L}x)(t), \quad t \in [0, \omega], \quad y(0) \geq x(0), \quad y'(0) \geq x'(0) \quad (1.3)$$

imply $y(t) \geq x(t)$ for $t \in [0, \omega]$.

Independently Azbelev [1], Beckenbach, Bellman [6] and Wilkins [37] established that (1.3) can be applied in a nonoscillation interval $[0, \omega]$ only. The general solution of equation (1.1) has the following representation

$$x(t) = \int_0^t C(t, s)f(s)ds + x_1(t)x_0 + x_2(t)x'_0, \quad (1.4)$$

where x_1 and x_2 are the solutions of the homogeneous equation

$$\mathcal{L}x(t) \equiv \ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(t) = 0, \quad t \in [0, \omega], \quad (1.5)$$

satisfying the initial conditions $x_1(0) = 1$, $x'_1(0) = 0$ and $x_2(0) = 0$, $x'_2(0) = 1$, respectively, x_0 and x'_0 are corresponding constants. The kernel $C(t, s)$ of the integral in solution's representation (1.4) is called *the Cauchy function* of equation (1.1). *The fundamental function* $X(t, s)$ of (1.1) is defined as follows: $X(t, s)$ for each fixed $s \geq 0$ as a function of t satisfies

$$\ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(t) = 0, \quad t \in [s, \omega], \quad (1.6)$$

$$x(s) = 0, \quad x'(s) = 1. \quad (1.7)$$

For equation (1.1) the Cauchy function and the fundamental function $X(t, s)$ coincide [5]. We assume that $X(t, s) = C(t, s) = 0$ for $0 \leq t < s$.

If the solution of the initial value problem $(\mathcal{L}x)(t) = 0$, $x(0) = 0$, $x'(0) = 1$ does not vanish at the point $t = \omega$, then the boundary value problem

$$\mathcal{L}x(t) \equiv \ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(t) = f(t), \quad t \in [0, \omega], \quad x(0) = 0, \quad x(\omega) = 0, \quad (1.8)$$

is uniquely solvable and its solution has the representation

$$x(t) = \int_0^\omega G(t, s)f(s)ds, \quad (1.9)$$

where the kernel of the integral representation $G(t, s)$ is called the Green's function of the problem (1.8). Differential inequality theorems are actually results on positivity or negativity of corresponding Green's functions.

The equivalence of nonoscillation and the unique solvability of the interpolation problems

$$\begin{aligned} \mathcal{L}x(t) &\equiv \ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(t) = f(t), \quad t \in [0, \omega], \\ x(t_1) &= 0, \quad x(t_2) = 0, \quad 0 \leq t_1 < t_2 \leq \omega, \end{aligned} \quad (1.10)$$

is obvious [27]. Let us say that the Green's function of problem (1.10) behaves regularly if

$$G(t, s)(t - t_1)(t - t_2) > 0, \quad t, s \in (0, \omega), \quad t \neq t_1, \quad t \neq t_2. \quad (1.11)$$

It was first proven in [10] that nonoscillation of the equation $\mathcal{L}x = 0$ on the interval $[0, \omega]$ is necessary and sufficient for the regular behavior of Green's functions of interpolation problem (1.10) (see also the well known paper by Levin [27]).

Let us denote by $C_{[0,\omega]}$ the space of continuous functions $x : [0, \omega] \rightarrow \mathbb{R}$ with the norm $\|x\| = \max_{t \in [0,\omega]} |x(t)|$, and by $D_{[0,\omega]}$ the linear space of functions $x : [0, \omega] \rightarrow \mathbb{R}$ with absolute continuous derivatives. Let $b(t) = b^+(t) - b^-(t)$, where $b^+(t) \geq 0$, $b^-(t) \geq 0$, and

$$\mathcal{L}_0 x(t) \equiv \ddot{x}(t) + a(t)\dot{x}(t) - b^-(t)x(t) = 0, \quad t \in [0, \omega]. \quad (1.12)$$

It was proven in [2] that the boundary value problem

$$\mathcal{L}_0 x(t) \equiv \ddot{x}(t) + a(t)\dot{x}(t) - b^-(t)x(t) = f(t), \quad t \in [0, \omega], \quad x(0) = 0, \quad x(\omega) = 0, \quad (1.13)$$

is uniquely solvable and its Green's function $G_0(t, s)$ is negative $G_0(t, s) < 0$ for $t, s \in (0, \omega)$. In the space $C_{[0,\omega]}$ let us define the integral operator $K : C_{[0,\omega]} \rightarrow C_{[0,\omega]}$ by the equality

$$(Kx)(t) = - \int_0^\omega G_0(t, s)b^+(s)x(s)ds. \quad (1.14)$$

Note that the operator K actually maps $C_{[0,\omega]}$ into $D_{[0,\omega]}$ due to the properties of Green's function $G_0(t, s)$. Hence the equation $x = Kx + g$, where $g(t) = \int_0^\omega G_0(t, s)f(s)ds$, is equivalent to the boundary value problem (1.8).

The above argument can be summarized in the form of the statement on six equivalences.

Theorem A [2]. *The following assertions are equivalent:*

- 1) *there is $v \in D_{[0,\omega]}$ such that $v(t) \geq 0$ and $(\mathcal{L}v)(t) \leq 0$ for $t \in [0, \omega]$ and $v(0) + v(\omega) - \int_0^\omega (\mathcal{L}v)(t)dt > 0$;*
- 2) *$C(t, s) > 0$ for $0 \leq s \leq t \leq \omega$;*
- 3) *problem (1.8) has a unique solution for each summable f and $G(t, s) < 0$ for $t, s \in (0, \omega)$;*
- 4) *each nontrivial solution of the homogeneous equation $\mathcal{L}x = 0$ has at most one zero on $[0, \omega]$;*
- 5) *the spectral radius of the operator $K : C_{[0,\omega]} \rightarrow C_{[0,\omega]}$ is less than one;*
- 6) *problem (1.10) has a unique solution for each summable f and its Green's function behaves regularly for $t, s \in (0, \omega)$.*

The Polia-Mammanna decomposition [30, 32] is a possibility for representation of the operator $\mathcal{L} : D_{[0,\omega]} \rightarrow L_{[0,\omega]}$ in the form of products of the first order differential operators

$$\mathcal{L} = h_2 \frac{d}{dt} h_1 \frac{d}{dt} h_0, \quad (1.15)$$

where the real valued functions h_i do not have zeros and are smooth enough. This decomposition is possible if and only if $[0, \omega]$ is a nonoscillation interval. In particular, representation (1.15) allows us to obtain the generalized Rolle's theorem [27]: *if the solution x of equation (1.1) has more than 3 zeros on the nonoscillation interval, then the function f has at least one zero on this interval.*

The theory of oscillatory kernels plays an important role in oscillation of mechanical systems [14]. The oscillatory kernel $G(t, s)$ is characterized by the inequalities

$$G(t, s) > 0, \quad \det |G(t_i, s_j)|_1^m \geq 0, \quad 0 < t_1 < \dots < t_m < \omega, \quad 0 < s_1 < \dots < s_m < \omega, \quad m = 1, 2, \dots \quad (1.16)$$

while for $t_i = s_i$ ($i = 1, 2, \dots, m$) the inequality in (1.16) has to be strict. In [14] it was demonstrated that actually the fact that the kernel is oscillatory means that the integral operator $G : L_{[0,\omega]} \rightarrow D_{[0,\omega]}$ of the form $(Gf)(t) = \int_0^\omega G(t,s)f(s)ds$ does not increase the number of sign's changes. If we consider the integral operator, where the kernel $G(t,s)$ is the Green's function of problem (1.8), then in this case the inverse operator $\mathcal{L} : D_{[0,\omega]} \rightarrow L_{[0,\omega]}$ defined on the functions satisfying the conditions $x(0) = 0$, $x(\omega) = 0$ should not decrease the number of sign's changes. Although a direct verification of an infinite number of inequalities (1.16) is possible only for very simple kernels and cannot be implemented in most interesting for applications cases, non-decreasing of the number of sign's changes for the integral operators with Green's functions as kernels can be checked through the Polia-Mammana decomposition and the generalized Rolle's theorem. This connection of oscillatory kernels and Polia-Mammana decomposition was discovered by M.G.Kreĭn.

Thus nonoscillation solves in many important cases the problem of checking oscillatory kernels. As a conclusion we note that each new nonoscillation result or test develops all these directions.

2 Preliminaries

Let us start with the following simple corollaries of Theorem A. Choosing $v(t) = \exp(\lambda t)$ in the first assertion of Theorem A, we obtain the following result.

Corollary 1 *If there exists such a real constant λ such that*

$$\lambda^2 + a(t)\lambda + b(t) \leq 0, \quad (2.1)$$

then for each ω assertion 2)-6) of Theorem A are true. If in addition $b(t) \geq \beta > 0$ and this $\lambda < 0$, then equation (1.1) is exponentially stable (the fundamental function has an exponential estimate).

Solving the equation

$$\lambda^2 + a(t)\lambda + b(t) = 0, \quad (2.2)$$

for all t , we get $\lambda_1(t) = -\frac{a(t)}{2} - \sqrt{\frac{a^2(t)}{4} - b(t)}$ and $\lambda_2(t) = -\frac{a(t)}{2} + \sqrt{\frac{a^2(t)}{4} - b(t)}$.

Corollary 2 [27] *If for sufficiently large $t \geq t_0$ there exist ν_0, ν_1 and ν_2 such that $\lambda_1(t)$ and $\lambda_2(t)$ are real functions and $\nu_0 \leq \lambda_1(t) \leq \nu_1 \leq \lambda_2(t) \leq \nu_2$, where $\nu_0 < \nu_1 < \nu_2$, then the fundamental system of equation (1.1) satisfies the inequalities*

$$c_i \exp(\nu_{i-1}t) \leq x_i(t) \leq d_i \exp(\nu_i t), \quad i = 1, 2 \quad (c_i, d_i > 0; t \geq t_0). \quad (2.3)$$

If in addition $\nu_2 < 0$, then equation (1.1) is exponentially stable.

In order to study stability properties, we consider the scalar differential equation of the second order (1.1) under the following conditions:

(a1) $a(t), b(t)$ are Lebesgue measurable and essentially bounded functions on $[0, \infty)$.

(a2) $f : [t_0, \infty) \rightarrow R$ is a Lebesgue measurable locally essentially bounded function.

Definition. Eq. (1.1) is *(uniformly) exponentially stable*, if the fundamental function $X(t, s)$ of (1.1) has an exponential estimate if there exist positive numbers $K > 0, \lambda > 0$, such that

$$|X(t, s)| \leq K e^{-\lambda(t-s)}, \quad t \geq s \geq 0. \quad (2.4)$$

Consider the equation

$$\ddot{x}(t) + a\dot{x}(t) + bx(t) = 0, \quad (2.5)$$

where $a > 0, b > 0$ are positive numbers. This equation is exponentially stable. Denote by $Y(t, s)$ the fundamental function of (2.5).

Lemma 1 Let $a > 0, b > 0$.

$$1) \text{ If } a^2 > 4b \text{ then } \int_0^t |Y(t, s)| ds \leq \frac{1}{b}, \quad \int_0^t |Y'_t(t, s)| ds \leq \frac{2a}{\sqrt{a^2 - 4b}(a - \sqrt{a^2 - 4b})}.$$

$$2) \text{ If } a^2 < 4b \text{ then } \int_0^t |Y(t, s)| ds \leq \frac{4}{a\sqrt{4b - a^2}}, \quad \int_0^t |Y'_t(t, s)| ds \leq \frac{2(a + \sqrt{4b - a^2})}{a\sqrt{4b - a^2}}.$$

$$3) \text{ If } a^2 = 4b \text{ then } \int_0^t |Y(t, s)| ds \leq \frac{1}{b}, \quad \int_0^t |Y'_t(t, s)| ds \leq \frac{2}{\sqrt{b}}.$$

Proof. If $a^2 > 4b$ then the characteristic equation $\lambda^2 + a\lambda + b = 0$ has two negative roots

$$\lambda_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}, \quad 0 \geq \lambda_1 > \lambda_2.$$

By simple calculation we have

$$0 < Y(t, s) = \frac{1}{\sqrt{a^2 - 4b}} (e^{\lambda_1(t-s)} - e^{\lambda_2(t-s)}).$$

Since $\frac{1}{\lambda_1} e^{\lambda_1 t} < \frac{1}{\lambda_2} e^{\lambda_2 t}$ then

$$\int_0^t Y(t, s) ds \leq \frac{1}{\sqrt{a^2 - 4b}} \left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1} \right) = \frac{1}{b}.$$

We have $Y'_t(t, s) = \frac{1}{\sqrt{a^2 - 4b}} (\lambda_1 e^{\lambda_1(t-s)} - \lambda_2 e^{\lambda_2(t-s)})$. Hence $|Y'_t(t, s)| \leq \frac{|\lambda_1| + |\lambda_2|}{\sqrt{a^2 - 4b}} e^{\lambda_1(t-s)}$

$$= \frac{a}{\sqrt{a^2 - 4b}} e^{\lambda_1(t-s)} \quad \text{and} \quad \int_0^t |Y'_t(t, s)| ds \leq \frac{2a}{\sqrt{a^2 - 4b}(a - \sqrt{a^2 - 4b})}.$$

If $a^2 - 4b < 0$, then the characteristic equation has two complex roots and the fundamental function has the form

$$Y(t, s) = \frac{2}{\sqrt{4b - a^2}} e^{-\frac{a}{2}(t-s)} \sin\left(\frac{\sqrt{4b - a^2}}{2}(t - s)\right).$$

Hence $|Y(t, s)| \leq \frac{2}{\sqrt{4b - a^2}} e^{-\frac{a}{2}(t-s)}$ and $\int_0^t |Y(t, s)| ds \leq \frac{4}{a\sqrt{4b - a^2}}$. The second inequality in 2) is proven in a similar way as the second inequality in 1).

If $a^2 = 4b$, then the characteristic equation has the double root $\lambda = -\frac{a}{2}$ and

$$Y(t, s) = (t - s) e^{-\frac{a}{2}(t-s)}.$$

Since $\int_0^\infty s e^{-as/2} = \frac{4}{a^2} = \frac{1}{b}$ then the first inequality in 3) holds. The second inequality in 3) is proven similarly to the previous cases. \square

Let us introduce some functional spaces on a semi-axis. Denote by $L_\infty[t_0, \infty)$ the space of all essentially bounded on $[t_0, \infty)$ functions and by $C[t_0, \infty)$ the space of all continuous bounded on $[t_0, \infty)$ functions with the supremum norm.

Lemma 2 [11, 18] *Suppose (a1)-(a2) hold and there exists $t_0 \geq 0$ such that for every $f \in L_\infty[t_0, \infty)$ both the solution x of the problem*

$$\ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(t) = f(t), \quad t \geq t_0,$$

$$x(t) = 0, \dot{x}(t) = 0, \quad t \leq t_0,$$

and its derivative \dot{x} belong to $C[t_0, \infty)$. Then equation (1.1) is exponentially stable.

3 Integro-differential equation

To obtain positiveness conditions for the fundamental function of equation (1.1) we consider first a similar problem for the following integro-differential equation

$$\dot{y}(t) + \int_0^t e^{-\int_s^t a(\xi) d\xi} b(s) y(s) ds = 0, \quad (3.1)$$

for which we assume that condition (a1) holds.

Together with (1.8) we consider for each $t_0 \geq 0$ the initial value problem

$$\dot{y}(t) + \int_{t_0}^t e^{-\int_s^t a(\xi) d\xi} b(s) y(s) ds = f(t), \quad (3.2)$$

$$y(t_0) = y_0. \quad (3.3)$$

We assume that condition (a2) holds for the function $f(t)$.

Definition. A locally absolutely continuous on $[t_0, \infty)$ function $y : \mathbb{R} \rightarrow \mathbb{R}$ is called a *solution* of problem (3.2), (3.3) if it satisfies equation (3.2) for almost every $t \in [t_0, \infty)$, and equality (3.3) for $t = t_0$.

Definition. For each $s \geq 0$ the solution $Y(t, s)$ of the problem

$$\dot{y}(t) + \int_s^t e^{-\int_s^t a(\xi) d\xi} b(\tau) y(\tau) d\tau = 0, \quad t > s, \quad y(s) = 1, \quad (3.4)$$

is called a *fundamental function* of equation (3.1).

We assume $Y(t, s) = 0, t < s$.

Lemma 3 [18] *Let (a1)-(a2) hold. Then there exists one and only one solution of problem (3.2), (3.3) that can be presented in the form*

$$y(t) = Y(t, t_0)y_0 + \int_{t_0}^t Y(t, s)f(s)ds. \quad (3.5)$$

Let us obtain conditions under which equation (3.1) has a positive solution. We remark that the theorem remains true if the zero initial point is replaced by any $t_0 \geq 0$.

In future we will apply the following result ([5], Theorem 1.2.1). Let $L[0, T]$ be the space of all integrable on $[0, T]$ functions with the norm $\|f\| = \int_0^T |f(s)|ds$.

Lemma 4 [5] *Suppose that a function $r(t, s)$ is measurable over the square $[a, b] \times [a, b]$, for almost every s the function $r(t, s)$ as a function of t has finite one-sided limits at each point t and there exists a function $v \in L[0, T]$ such that $|r(\cdot, s)| \leq v(\cdot)$. Then the integral operator*

$$(Ky)(t) = \int_a^b r(t, s)y(s)ds$$

maps $L[0, T]$ to $L[0, T]$ and is a compact operator in this space.

Theorem 1 *Suppose $a(t) \geq 0, b(t) \geq 0$. Then the following conditions are equivalent for equation (3.1).*

1) *There exists a positive solution of the inequality*

$$\dot{y}(t) + \int_0^t e^{-\int_s^t a(\xi) d\xi} b(s)y(s)ds \leq 0, \quad t \geq 0. \quad (3.6)$$

2) *There exists a locally essentially bounded function $u(t) \geq 0$ such that*

$$u(t) \geq \int_0^t e^{-\int_s^t [a(\xi) - u(\xi)] d\xi} b(s)ds, \quad t \geq 0. \quad (3.7)$$

3) $Y(t, s) > 0, t \geq s \geq 0$.

4) *There exists a positive solution of equation (1.8) for $t \geq 0$.*

Proof. 1) \implies 2). Suppose $y(t) > 0$ is a solution of (3.6). Hence on any bounded interval $y(t) \geq \alpha > 0$, then $u(t) = -\frac{\dot{y}(t)}{y(t)} \geq 0$ and is an essentially locally bounded function. We also have

$$y(t) = y(0)e^{-\int_0^t u(\xi)d\xi}, \quad \dot{y}(t) = -y(0)u(t)e^{-\int_0^t u(\xi)d\xi}.$$

Substituting y, \dot{y} in (3.6) we have

$$\begin{aligned} & \dot{y}(t) + \int_0^t e^{-\int_s^t a(\xi)d\xi} b(s)y(s)ds \\ &= -y(0)u(t)e^{-\int_0^t u(\xi)d\xi} + y(0) \int_0^t e^{-\int_s^t a(\xi)d\xi} b(s)e^{-\int_0^s u(\xi)d\xi} ds \\ &= -y(0)e^{-\int_0^t u(\xi)d\xi} \left[u(t) - \int_0^t e^{-\int_s^t [a(\xi)-u(\xi)]d\xi} b(s)ds \right] \leq 0. \end{aligned}$$

Hence $u(t) \geq 0$ is a solution of inequality (3.7).

2) \implies 3). Consider the nonhomogeneous equation (3.6)

$$\dot{y}(t) + \int_0^t e^{-\int_s^t a(\xi)d\xi} b(s)y(s)ds = f(t), \quad y(0) = 0. \quad (3.8)$$

A solution of (3.8) will be sought in the form

$$y(t) = \int_0^t e^{-\int_s^t u(\xi)d\xi} z(s)ds, \quad \dot{y}(t) = z(t) - u(t) \int_0^t e^{-\int_s^t u(\xi)d\xi} z(s)ds, \quad (3.9)$$

so $\dot{y}(t) + u(t)y(t) = z(t), y(0) = 0$.

After substituting (3.9) into (3.8) we have

$$z(t) - u(t) \int_0^t e^{-\int_s^t u(\xi)d\xi} z(s)ds + \int_0^t e^{-\int_s^t a(\xi)d\xi} b(s) \int_0^s e^{-\int_\tau^s u(\xi)d\xi} z(\tau)d\tau ds = f(t). \quad (3.10)$$

Since

$$\begin{aligned} & \int_0^t \left[e^{-\int_s^t a(\xi)d\xi} b(s) \int_0^s e^{-\int_\tau^s u(\xi)d\xi} z(\tau)d\tau \right] ds \\ &= \int_0^t \left[\int_\tau^t e^{-\int_s^t a(\xi)d\xi} b(s) e^{-\int_\tau^s u(\xi)d\xi} ds \right] z(\tau)d\tau \\ &= \int_0^t e^{-\int_\tau^t u(\xi)d\xi} \left[\int_\tau^t e^{-\int_s^t (a(\xi)-u(\xi))d\xi} b(s)ds \right] z(\tau)d\tau, \end{aligned}$$

then equation (3.10) can be rewritten as

$$z(t) - \int_0^t e^{-\int_\tau^t u(\xi)d\xi} \left[u(t) - \int_\tau^t e^{-\int_s^t (a(\xi)-u(\xi))d\xi} b(s)ds \right] z(\tau)d\tau = f(t). \quad (3.11)$$

On every finite interval $[0, T]$ equation (3.11) has the form

$$z - Hz = f, \quad t \in [0, T]. \quad (3.12)$$

Operator $H : L[0, T] \rightarrow L[0, T]$ is bounded. In order to show that this operator is compact we apply Lemma 4. Operator H can be rewritten in the form $H = PH_1 - H_2$, where

$$(Pz)(t) = u(t)z(t), \quad (H_1z)(t) = \int_0^t e^{-\int_\tau^t a(\xi)d\xi} z(\tau)d\tau,$$

$$(H_2z)(t) = \int_0^t \left[e^{-\int_\tau^t a(\xi)d\xi} \int_\tau^t e^{-\int_s^t (a(\xi)-u(\xi))d\xi} b(s)ds \right] z(\tau)d\tau.$$

It is easy to see that for operators H_1, H_2 all conditions of Lemma 5 hold. Then these operators are compact. Operator P is bounded operator, hence operator H is a compact Volterra integral operator with spectral radius $r(T) = 0$ [34]. Hence for the solution of equation (3.12) we have $z = (I - H)^{-1}f$, where I is the identity operator.

Since

$$u(t) - \int_\tau^t e^{-\int_s^t (a(\xi)-u(\xi))d\xi} b(s)ds \geq u(t) - \int_0^t e^{-\int_s^t (a(\xi)-u(\xi))d\xi} b(s)ds \geq 0,$$

then H is a positive operator. Hence $(I - H)^{-1} = 1 + H + H^2 + H^3 + \dots$ is also a positive operator.

Suppose now that in the equation (3.12) we have $f(t) \geq 0$. Then for the solution of (3.12) we have $z(t) \geq 0$. Equality (1.16) implies that for every right-hand side $f(t) \geq 0$ the solution of equation (3.8) $y(t) \geq 0$. But $y(t) = \int_0^t Y(t, s)f(s)ds$. Hence $Y(t, s) \geq 0$, $0 \leq s \leq t \leq T$. Since $T > 0$ is an arbitrary number then $Y(t, s) \geq 0$, $0 \leq s \leq t < \infty$. We only have to prove that the strong inequality for $Y(t, s) > 0$ holds.

After substituting $y(t) = e^{-\int_0^t u(\xi)d\xi}$ in the left-hand side of equation (3.8) we see that this function is a solution of (3.8) with $f(t) < 0$. By the solution representation formula (3.5) we have

$$y(t) = Y(t, 0) + \int_0^t Y(t, s)f(s)ds.$$

Hence $Y(t, 0) \geq y(t) > 0$. The general case $Y(t, s) > 0$ is considered similarly.

Implications 3) \implies 4) and 4) \implies 1) are evident. □

Remark. Nonoscillation conditions for general integro-differential equations with a bounded memory were obtained in [7]. However, these results are not applicable to equation (3.1). Nonoscillation results for integro-differential equation can also be found in [12].

4 Positive Solutions

The following lemma gives a connection between equations (1.1) and (3.1).

Lemma 5 *Denote by $x_1(t), x_2(t)$ and $X(t, s)$ the fundamental system and the fundamental function of (1.1), respectively, by $Y(t, s)$ the fundamental function of (1.8). Then*

$$x_1(t) = Y(t, 0), \quad x_2(t) = \int_0^t Y(t, \tau)e^{-\int_0^\tau a(\xi)d\xi}d\tau, \quad X(t, s) = \int_s^t Y(t, \tau)e^{-\int_s^\tau a(\xi)d\xi}d\tau.$$

Proof. For the solution of equation (1.1) we have

$$\dot{x}(t) = e^{-\int_0^t a(\xi)d\xi} \dot{x}(0) - \int_0^t e^{-\int_s^t a(\xi)d\xi} b(s)x(s)ds.$$

Hence

$$\dot{x}(t) + \int_0^t e^{-\int_s^t a(\xi)d\xi} b(s)x(s)ds = e^{-\int_0^t a(\xi)d\xi} \dot{x}(0)$$

and

$$x(t) = Y(t, 0)x(0) + \left[\int_0^t Y(t, \tau) e^{-\int_0^\tau a(\xi)d\xi} d\tau \right] \dot{x}(0).$$

But for the solution of (1.1) we have another representation:

$$x(t) = x_1(t)x(0) + x_2(t)\dot{x}(0).$$

The equalities for the fundamental system of (1.1) are proven. Since $X(t, s) = x_2(t, s)$, then the proof of the equality for $X(t, s)$ is similar. \square

Corollary 3 *If the fundamental function $Y(t, s)$ of (3.1) is positive then the fundamental system and the fundamental function of (1.1) are positive.*

Corollary 4 *Suppose $a(t) \geq 0, b(t) \geq 0$, and the fundamental function of (1.8) is positive. Then*

$$0 \leq \int_{t_0}^t X(t, s)b(s)ds \leq 1, \quad (4.1)$$

where $X(t, s)$ is the fundamental function of (1.1).

Proof. The function $x(t) \equiv 1$ is the solution of (1.2) with $f(t) = b(t)$. By the solution representation formula we have

$$1 = x_1(t) + \int_{t_0}^t X(t, s)b(s)ds.$$

Corollary 3 implies $x_1(t) > 0, X(t, s) > 0$. Hence the inequality (4.1) is valid. \square

Together with (1.1) consider the following equation

$$\ddot{x}(t) + a_1(t)\dot{x}(t) + b_1(t)x(t) = 0, \quad (4.2)$$

where for $a_1(t), b_1(t)$ condition (a1) holds.

Corollary 5 *Suppose $a_1(t) \geq a(t) \geq 0, b(t) \geq b_1(t) \geq 0$, and equation (3.1) has a positive solution. Then the fundamental function and the fundamental system of (4.2) are positive.*

Proof. If (3.1) has a positive solution, then inequality (3.7) has a nonnegative solution $u(t) \geq 0$. This function is a nonnegative solution of inequality (3.7) where $a(t)$ and $b(t)$ are replaced by $a_1(t)$ and $b_1(t)$. Corollary 3 implies this corollary. \square

Theorem 2 Suppose $a_1(t) \geq a(t) \geq 0, b(t) \geq b_1(t), b(t) \geq 0$. If the fundamental function of (3.1) is positive, then the fundamental function and the fundamental system of (4.2) are positive.

Proof. Consider the equation

$$\dot{y}(t) + \int_0^t e^{-\int_s^t a_1(\xi) d\xi} b_1(s) y(s) ds = f(t), \quad y(0) = 0, \quad (4.3)$$

which can be rewritten in the form

$$\begin{aligned} & \dot{y}(t) + \int_0^t e^{-\int_s^t a(\xi) d\xi} b(s) y(s) ds \\ &= \int_0^t \left(e^{-\int_s^t a(\xi) d\xi} - e^{-\int_s^t a_1(\xi) d\xi} \right) b(s) y(s) ds \\ &+ \int_0^t e^{-\int_s^t a_1(\xi) d\xi} (b(s) - b_1(s)) y(s) ds + f(t), \quad y(0) = 0. \end{aligned} \quad (4.4)$$

Hence (4.4) is equivalent to the equation

$$\begin{aligned} y(t) &= \int_0^t Y(t, s) \int_0^s \left(e^{-\int_\tau^s a(\xi) d\xi} - e^{-\int_\tau^s a_1(\xi) d\xi} \right) b(\tau) y(\tau) d\tau ds \\ &+ \int_0^t Y(t, s) \int_0^s e^{-\int_\tau^s a_1(\xi) d\xi} (b(\tau) - b_1(\tau)) y(\tau) d\tau ds + g(t), \end{aligned} \quad (4.5)$$

where $g(t) = \int_0^t Y(t, s) f(s) ds \geq 0$.

Equation (4.5) has the form $y = Hy + g$, where H is a positive compact Volterra operator in the space $L[0, T]$ for every $T > 0$. Hence the spectral radius $r(H) = 0$ in the space $L[0, T]$ and then for the solution of (4.5) we have $y(t) = ((I - T)^{-1}g)(t) \geq 0$. The solution of the equation (4.3) has the form $y(t) = \int_0^t Y_1(t, s) f(s) ds$, where $Y_1(t, s)$ is the fundamental function of the equation

$$\dot{y}(t) + \int_0^t e^{-\int_s^t a_1(\xi) d\xi} b_1(s) y(s) ds = 0.$$

We obtained that for every $f(t) \geq 0$ the solution of (4.3) is also nonnegative. Hence the inequality $Y_1(t, s) \geq 0$ holds. Now, let us prove the strict inequality $Y_1(t, s) > 0$.

The function $Y_1(t, 0)$ is the solution of the equation

$$\dot{y}(t) + \int_0^t e^{-\int_s^t a_1(\xi) d\xi} b_1(s) y(s) ds = 0, \quad y(0) = 1. \quad (4.6)$$

After rewriting (4.6) in the form

$$\begin{aligned} & \dot{y}(t) + \int_0^t e^{-\int_s^t a(\xi) d\xi} b(s) y(s) ds \\ &= \int_0^t \left(e^{-\int_s^t a(\xi) d\xi} - e^{-\int_s^t a_1(\xi) d\xi} \right) b(s) y(s) ds \\ &+ \int_0^t e^{-\int_s^t a_1(\xi) d\xi} (b(s) - b_1(s)) y(s) ds, \quad y(0) = 1, \end{aligned} \quad (4.7)$$

equation (4.7) has the form

$$\dot{y}(t) + \int_0^t e^{-\int_s^t a(\xi) d\xi} b(s) y(s) ds = f(t), y(0) = 1,$$

where $f(t) \geq 0$. Hence

$$Y_1(t, 0) = Y(t, 0) + \int_0^t Y(t, s) f(s) ds.$$

Then $Y_1(t, 0) \geq Y(t, 0) > 0$.

The general case $Y_1(t, s) > 0$ is considered similarly. Now, Lemma 6 implies the statement of the theorem. \square

Remark. The theorem remains true if we replace the zero initial point 0 by any $t_0 > 0$.

Denote $b^+ = \max\{b, 0\}$.

Corollary 6 Suppose $a(t) \geq 0$ and the fundamental function of the equation

$$\dot{y}(t) + \int_0^t e^{-\int_s^t a(\xi) d\xi} b^+(s) y(s) ds = 0$$

is positive. Then the fundamental function and the fundamental system of equation (1.1) are positive.

Now let us proceed to explicit sufficient conditions when equation (1.1) has a positive solution.

Theorem 3 Suppose at least one of the following conditions holds

- 1) $a(t) \geq \int_{t_0}^t b^+(s) ds$,
- 2) there exists $\lambda > 0$ such that $a(t) \geq \lambda b^+(t) + \frac{1}{\lambda}$, $t \geq t_0$,
- 3) $a(t) \geq 0$ and there exists $\lambda > 0$ such that $\int_{t_0}^t e^{-\int_s^t (a(\xi) - \lambda) d\xi} b^+(s) ds \leq \lambda$, $t \geq t_0$.

Then the fundamental function and the fundamental system of (1.1) are positive for $t \geq t_0$.

Proof. It is sufficient to prove the theorem for the case $b(t) \geq 0$.

- 1) The function $u(t) = a(t)$ is a solution of inequality (3.7).
- 2) The function $u(t) = a(t) - \lambda b(t)$ is a solution of inequality (3.7).
- 3) The function $u(t) = \lambda$ is a solution of inequality (3.7). \square

Corollary 7 Suppose $a(t) \geq 0$ and at least one of the following conditions holds

- 1) $a(t) \geq a > 0$, $\int_0^\infty b^+(s) ds < \infty$,
- 2) $a^2(t) \geq 4B$, where $B = \limsup_{t \rightarrow \infty} b^+(t)$,
- 3) $\inf_{\lambda > 0} \limsup_{t \rightarrow \infty} \frac{1}{\lambda} \int_0^t e^{-\int_s^t (a(\xi) - \lambda) d\xi} b^+(s) ds < 1$.

Then there exists $t_0 \geq 0$ such that the fundamental function and the fundamental system of (1.1) are positive for $t \geq t_0$.

Proof. The proof of 1) and 3) is evident. To prove 2) we assume $\lambda = 1/\sqrt{B}$. \square

Now let us compare solutions of the same equation with different right hand sides and initial conditions, as well as of different equations.

Theorem 4 Suppose $a(t) \geq 0, b(t) \geq 0$, the fundamental function of (3.1) is positive. Denote by $x(t)$ the solution of the problem

$$\ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(t) = f(t), \quad t \geq t_0,$$

$$x(t_0) = x_0, \quad x'(t_0) = x'_0,$$

by $v(t)$ the solution of the above problem, where $f(t), x_0, x'_0$ are replaced with $f_1(t), v_0, v'_0$. If $f(t) \geq f_1(t), x_0 \geq v_0, x'_0 \geq v'_0$, then $x(t) \geq v(t)$.

Proof. By Corollary 1 the fundamental system and the fundamental function of (1.1) are positive. The statement of the theorem follows from solution representation formula (1.4). \square

Theorem 5 Suppose $a_1(t) \geq a(t) \geq 0, b(t) \geq b_1(t) \geq 0$, the fundamental function of (3.1) is positive. Denote by $x_1(t), x_2(t), X(t, s)$ the fundamental system and the fundamental function of (1.1); by $v_1(t), v_2(t), V(t, s)$ the fundamental system and the fundamental function of (4.2). Then $x_1(t) \leq v_1(t), x_2(t) \leq v_2(t), X(t, s) \leq V(t, s)$.

Proof. Denote by $Y(t, s)$ the fundamental function of (1.8), by $Y_1(t, s)$ the fundamental function of (3.1) where $a(t), b(t)$ is replaced by $a_1(t), b_1(t)$, respectively. By Corollary 1 it is sufficient to prove that $Y_1(t, s) \geq Y(t, s)$.

The function $Y_1(t, 0)$ is the solution of the equation

$$\dot{y}(t) + \int_0^t e^{-\int_s^t a_1(\xi)d\xi} b_1(s)y(s)ds = 0, \quad y(0) = 1, \quad (4.8)$$

which can be rewritten equation in the form

$$\begin{aligned} \dot{y}(t) + \int_0^t e^{-\int_s^t a_1(\xi)d\xi} b(s)y(s)ds &= \int_0^t e^{-\int_s^t a(\xi)d\xi} (b(s) - b_1(s))y(s)ds \\ &+ \int_0^t \left(e^{-\int_s^t a(\xi)d\xi} - e^{-\int_s^t a_1(\xi)d\xi} \right) b_1(s)y(s)ds, \quad y(0) = 1. \end{aligned} \quad (4.9)$$

By formula (3.5) for the solution of (4.9) we have

$$\begin{aligned} Y_1(t, 0) &= Y(t, 0) + \int_0^t Y(t, s) \int_0^s e^{-\int_\tau^s a(\xi)d\xi} (b(\tau) - b_1(\tau))y(\tau)d\tau ds \\ &+ \int_0^t Y(t, s) \int_0^s \left(e^{-\int_\tau^s a(\xi)d\xi} - e^{-\int_\tau^s a_1(\xi)d\xi} \right) b_1(\tau)y(\tau)d\tau ds. \end{aligned}$$

Since $a(t) - a_1(t) \geq 0, b(t) - b_1(t) \geq 0$, then $Y_1(t, 0) \geq Y(t, 0)$. The inequality $Y_1(t, s) \geq Y(t, s)$ is obtained similarly. \square

Corollary 8 *Suppose*

$$a(t) \equiv a > 0, \quad b(t) \geq b > 0, \quad a^2 - 4b \geq 0.$$

Then the fundamental function $X(t, s)$ of (1.1) is positive and for this function we have the following estimation

$$\int_0^t X(t, s) ds \leq \frac{1}{b}.$$

Proof. By Theorem 5 we have $0 < X(t, s) \leq V(t, s)$, where $V(t, s)$ is the fundamental function of equation (2.5). Application of Lemma 2 (parts 1 and 3) completes the proof. \square

5 Stability

Theorem 6 *Suppose $a(t) \geq \alpha > 0, b(t) \geq \beta > 0$. If the fundamental function of (1.1) is positive then equation (1.1) is exponentially stable.*

Proof. Consider (1.1) with initial conditions $x(0) = 0, \dot{x}(0) = 0$. Suppose $f \in L_\infty[0, \infty)$. For the solution of this problem we have $x(t) = \int_0^t X(t, s)f(s)ds$. By Corollary 4

$$|x(t)| \leq \int_0^t X(t, s)|f(s)|ds = \int_0^t X(t, s)b(s)\frac{|f(s)|}{b(s)}ds \leq \frac{\|f\|}{\beta} \int_0^t X(t, s)b(s)ds \leq \frac{\|f\|}{\beta},$$

where $\|\cdot\|$ is the sup-norm in $C[0, \infty)$. Then $x \in C[0, \infty)$. We have

$\dot{x}(t) = \int_0^t e^{-\int_s^t a(\xi)d\xi}b(s)x(s)ds + \int_0^t e^{-\int_s^t a(\xi)d\xi}b(s)f(s)ds$, hence $\dot{x} \in C[0, \infty)$. By Lemma 3 equation (1.1) is exponentially stable. \square

Corollary 9 *Suppose $a(t) \geq \alpha > 0, b(t) \geq \beta > 0$ and at least one of the conditions of Corollary 7 holds. Then equation (1.1) is exponentially stable.*

Denote

$$\alpha = \liminf_{t \rightarrow \infty} a(t), \beta = \liminf_{t \rightarrow \infty} b(t), B = \limsup_{t \rightarrow \infty} b(t).$$

Theorem 7 *Suppose $0 < \beta \leq B < \frac{1}{2}\alpha^2$. Then equation (1.1) is exponentially stable.*

Proof. Without loss of generality we can assume that $b(t) \geq \beta$ for any $t \geq 0$ and there exists $\epsilon > 0$ such that

$$\frac{\epsilon}{4}\alpha^2 < b(t) < \left(\frac{1}{2} - \frac{\epsilon}{4}\right)\alpha^2. \quad (5.1)$$

Consider (1.1) with initial conditions $x(0) = 0, \dot{x}(0) = 0$. Suppose $f \in L_\infty[0, \infty)$ and let us prove that $x \in C[0, \infty)$.

Denote by $X_0(t, s)$ the fundamental function of the equation

$$\ddot{x}(t) + a(t)\dot{x}(t) + \frac{1}{4}\alpha^2 x(t) = 0. \quad (5.2)$$

By Corollary 7(2), $X_0(t, s) > 0, t \geq s \geq 0$. Equation (1.1) can be rewritten in the form in the form

$$\ddot{x}(t) + a(t)\dot{x}(t) + \frac{1}{4}\alpha^2 x(t) + \left(b(t) - \frac{1}{4}\alpha^2\right) x(t) = f(t), \quad (5.3)$$

with the initial conditions $x(0) = 0, \dot{x}(0) = 0$. Equation (5.3) is equivalent to

$$x(t) + \int_0^t X_0(t, s) \left(b(s) - \frac{1}{4}\alpha^2\right) x(s) ds = g(t), \quad (5.4)$$

where $g(t) = \int_0^t X_0(t, s) f(s) ds$. By Theorem 6 equation (5.2) is exponentially stable hence $X_0(t, s)$ has an exponential estimation. Then $g \in L_\infty[0, \infty)$.

Equation (5.4) has the form $x + Px = g$. Corollary 2 implies

$$\begin{aligned} |(Px)(t)| &\leq \int_0^t X_0(t, s) \left|b(s) - \frac{1}{4}\alpha^2\right| |x(s)| ds \\ &\leq \int_0^t X_0(t, s) \frac{1}{4}\alpha^2 \frac{|b(s) - \frac{1}{4}\alpha^2|}{\frac{1}{4}\alpha^2} ds \|x\| \\ &\leq \frac{\sup_{t \geq 0} |b(t) - \frac{1}{4}\alpha^2|}{\frac{1}{4}\alpha^2} \|x\|. \end{aligned}$$

Inequality $\frac{\sup_{t \geq 0} |b(t) - \frac{1}{4}\alpha^2|}{\frac{1}{4}\alpha^2} \leq 1 - \epsilon$ is equivalent to (5.1). Hence $\|P\| < 1$, where the norm is in $C[0, \infty)$. Then $x \in C[0, \infty)$ for a solution of (5.4) and therefore for a solution of (1.1). Similar to the proof of Theorem 6 we have $\dot{x} \in C[0, \infty)$. By Lemma 3 equation (1.1) is exponentially stable. \square

Example 1. Consider the following equation

$$\ddot{x}(t) + a\dot{x}(t) + (1 + 0.99 \sin t)x(t) = 0. \quad (5.5)$$

If $a > \sqrt{3.98}$ then by Theorem 7 equation (5.5) is exponentially stable.

In all previous results we obtain stability conditions for equations with a positive fundamental function and for "small" perturbations of such equations. Below we will give stability conditions for equations without any positiveness assumptions.

Theorem 8 Suppose $a(t) \geq \alpha > 0, b(t) \geq 0$ and there exist $A > 0, B > 0$ such that the following condition holds

$$\|a(t) - A\| \left\| \frac{b}{a} \right\| + \|b(t) - B\| < \begin{cases} B, & A^2 \geq 4B, \\ \frac{A\sqrt{4B-A^2}}{4}, & A^2 < 4B, \end{cases} \quad (5.6)$$

where $\|\cdot\|$ is the norm in the space $L_\infty[t_0, \infty)$ for some $t_0 \geq 0$. Then equation (1.1) is exponentially stable.

Proof. Without loss of generality we can assume $t_0 = 0$. Consider equation (1.1) with the initial conditions $x(0) = 0, \dot{x}(0) = 0$. Suppose $f \in L_\infty[0, \infty)$. Let us prove that $x, \dot{x} \in C[0, \infty)$. To this end, rewrite equation (1.1) in the form

$$\ddot{x}(t) + A\dot{x}(t) + Bx(t) + (a(t) - A)\dot{x}(t) + (b(t) - B)x(t) = f(t), \quad x(0) = \dot{x}(0) = 0. \quad (5.7)$$

Hence

$$x(t) + \int_0^t Y(t, s) [(a(s) - A)\dot{x}(s) + (b(s) - B)x(s)] ds = f_1(t), \quad (5.8)$$

where $Y(t, s)$ is the fundamental function of

$$\ddot{x}(t) + A\dot{x}(t) + Bx(t) = 0, \quad (5.9)$$

$f_1(t) = \int_0^t Y(t, s)f(s)ds$. Equation (5.9) is exponentially stable thus $f_1 \in L_\infty[0, \infty)$.

Equation (1.1) can be rewritten in a different form

$$\dot{x}(t) + \int_0^t e^{-\int_s^t a(\xi)d\xi} b(s)x(s)ds = r(t), \quad t \geq 0, \quad (5.10)$$

where $r(t) = \int_0^t e^{-\int_s^t a(\xi)d\xi} f(s)ds$. Since $a(t) \geq \alpha > 0$, then $r \in L_\infty[0, \infty)$. Substituting $\dot{x}(t)$ from (5.10) to (5.8) we have

$$x(t) - \int_0^t Y(t, s) \left[(a(s) - A) \int_0^s e^{-\int_\tau^s a(\xi)d\xi} b(\tau)x(\tau)d\tau - (b(s) - B)x(s) \right] ds = f_2(t), \quad (5.11)$$

where $f_2(t) = f_1(t) + \int_0^t Y(t, s)r(s)ds$. Evidently $f_2 \in L_\infty[0, \infty)$.

Equation (5.11) has the form $x - Hx = f_2$, where

$$\begin{aligned} \|(Hx)(t)\| &\leq \int_0^t Y(t, s) \left[|a(s) - A| \int_0^s e^{-\int_\tau^s a(\xi)d\xi} a(\tau) \frac{b(\tau)}{a(\tau)} d\tau + |b(s) - B| \right] ds \|x\| \\ &\leq K \left(\|a(t) - A\| \left\| \frac{b}{a} \right\| + \|b(t) - B\| \right) \|x\|, \end{aligned}$$

where by Lemma 2 the constant K is

$$K = \sup_{t \geq 0} \int_0^t |Y(t, s)| ds = \begin{cases} \frac{1}{B}, & A^2 \geq 4B, \\ \frac{4}{A\sqrt{4B-A^2}}, & A^2 < 4B. \end{cases}$$

Then $\|H\| < 1$, hence $x \in C[0, \infty)$ for the solution x of (5.11) and therefore of (1.1). As in the proof of Theorem 7, $\dot{x} \in C[0, \infty)$. By Lemma 3 equation (1.1) is exponentially stable. \square

Corollary 10 Suppose there exist $a > 0, B > 0$ such that

$$B - \frac{a\sqrt{4B-a^2}}{4} < m \leq M < B + \frac{a\sqrt{4B-a^2}}{4}, \quad (5.12)$$

where $m = \liminf_{t \rightarrow \infty} b(t) > 0$, $M = \limsup_{t \rightarrow \infty} b(t)$. Then the equation

$$\ddot{x}(t) + a\dot{x}(t) + b(t)x(t) = 0 \quad (5.13)$$

is exponentially stable.

Proof follows from the second inequality (4.5) if we let $A = a$. \square

Application of the first inequality (4.5) to equation (5.13) gives the same stability conditions which were obtained by application of Theorem 7.

Example 2. Consider the equation

$$\ddot{x}(t) + (10 + \alpha(t) \sin t)\dot{x}(t) + (26 + \beta(t) \cos t)x(t) = 0, \quad (5.14)$$

where $\alpha(t), \beta(t)$ are measurable functions, such that $|\alpha(t)| \leq 1, |\beta(t)| \leq 1$.

If we take $A = 10, B = 26$, then all condition of Theorem 8 hold. Then equation (5.14) is exponentially stable.

We will obtain new stability conditions using the derivative of the fundamental function of comparison equations.

Theorem 9 Suppose $a(t) \geq \alpha > 0, b(t) \geq 0$ and there exist $a > 0, b > 0$ such that at least one of the following conditions holds:

- 1) $a^2 > 4b$, $\|a(t) - a\| \frac{2a}{\sqrt{a^2 - 4b}(a - \sqrt{a^2 - 4b})} + \|b(t) - b\| \frac{1}{b} < 1$.
- 2) $a^2 < 4b$, $\|a(t) - a\| \frac{2(a + \sqrt{4b - a^2})}{a\sqrt{4b - a^2}} + \|b(t) - b\| \frac{4}{a\sqrt{4b - a^2}} < 1$.
- 3) $a^2 = 4b$, $\|a(t) - a\| \frac{2}{\sqrt{b}} + \|b(t) - b\| \frac{1}{b} < 1$,

where $\|\cdot\|$ is the norm in the space $L_\infty[t_0, \infty)$ for some $t_0 \geq 0$. Then equation (1.1) is exponentially stable.

Proof. Without loss of generality we can assume $t_0 = 0$. Suppose $x(t)$ is a solution of (1.1) with initial conditions $x(0) = x'(0) = 0$. Denote $z(t) = \ddot{x}(t) + a\dot{x}(t) + bx(t)$, $Y(t, s)$ is the fundamental function of the equation $\ddot{x}(t) + a\dot{x}(t) + bx(t) = 0$. Then

$$x(t) = \int_0^t Y(t, s)z(s)ds, \quad \dot{x}(t) = \int_0^t Y'_t(t, s)z(s)ds. \quad (5.15)$$

Equation (1.1) is equivalent to the equation

$$\ddot{x}(t) + a\dot{x}(t) + b(t)x(t) + (a(t) - a)\dot{x}(t) + (b(t) - b)x(t) = f(t). \quad (5.16)$$

After substituting $z(t)$ and (5.15) into (5.16) we have the following equation

$$z(t) + (a(t) - a) \int_0^t Y'_t(t, s)z(s)ds + (b(t) - b) \int_0^t Y(t, s)z(s)ds = f(t). \quad (5.17)$$

Equation (5.17) has the form $z + Hz = f$. For the norm of the operator H in the space $L_\infty[0, \infty)$ we have

$$\|H\| \leq \|a(t) - a\| \left\| \int_0^t |Y'_t(t, s)| ds \right\| + \|b(t) - b\| \left\| \int_0^t |Y(t, s)| ds \right\|.$$

By Lemma 2 conditions 1)-3) of the theorem imply $\|H\| < 1$. Hence for the solution of (5.17) we have $z \in L_\infty[0, \infty)$. Equation $\ddot{x}(t) + a\dot{x}(t) + bx(t) = 0$ is exponentially stable. Equalities (5.15) imply that the solution of (1.1) and its derivative are bounded functions. Then by Lemma 3 equation (1.1) is exponentially stable. \square

Example 3. Consider the following equation

$$\ddot{x}(t) + \dot{x}(t) + (b + \sin t)x(t) = 0. \quad (5.18)$$

If $b > 4.25$ then by Theorem 9(2) ($a = 1$, b is the same as in the equation) equation (5.18) is exponentially stable. Theorem 8 gives the same result, if we take $a = 1, B = b$. Theorem 7 does not give any stability condition for this equation.

6 Zones of Lyapunov's Stability

In the following, let us assume that $a(t)$ is an absolutely continuous function.

It is known that the substitution

$$x(t) = z(t) \exp \left\{ -\frac{1}{2} \int_0^t a(s) ds \right\} \quad (6.1)$$

transforms the homogeneous equation

$$\mathcal{L}x(t) \equiv \ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(t) = 0, \quad t \in [0, \infty), \quad (6.2)$$

into the equation

$$\ddot{z}(t) + p(t)z(t) = 0, \quad t \in [0, \infty), \quad (6.3)$$

where

$$p(t) = b(t) - \frac{a^2(t)}{4} - \frac{a'(t)}{2}. \quad (6.4)$$

Obviously coefficient $p(t)$ can be presented as $p(t) = p^+(t) - p^-(t)$, where $p^+(t) \geq 0$ and $p^-(t) \geq 0$.

Consider now equation (6.2) with an ω -periodic coefficients $a(t)$ and $b(t)$.

It is known from the works of the well known mathematicians Zhukovskii [40], Kreĭn [26] and Yakubovich [38] that there is a deep connection between the problem of the Lyapunov's stability and the nonoscillation intervals. We propose the following statement.

Theorem 10 Assume that $a(t + \omega) = a(t)$, $b(t + \omega) = b(t)$ for $t \in [0, \infty)$ and

$$\int_0^\omega p(t)dt > 0, \quad (6.5)$$

where $p(t)$ is defined in (6.4), and at least one of the following three conditions holds:

- 1) $a(t) \geq \int_0^t b^+(s)ds$ for $t \in [0, \omega]$;
- 2) $a(t) \geq 0$ and there exists $\lambda > 0$ such that $\int_0^t \exp \left\{ - \int_s^t (a(\xi) - \lambda)d\xi \right\} b^+(s)ds \leq \lambda$ for $t \in [0, \omega]$;
- 3) $\int_0^\omega p^+(t)dt \leq \frac{4}{\omega}$.

Then all solutions of homogeneous equation (6.2) tend to zero when $t \rightarrow \infty$, if $\int_0^\omega a(t)dt > 0$, and all solutions are bounded if $\int_0^\omega a(t)dt = 0$.

Proof. It is known [27] that if $[0, \omega]$ is a nonoscillation interval for (6.2), where ω is the period of the coefficient $p(t)$, then condition (6.5) guarantees that all solutions of equation (6.2) are bounded. Each of the conditions 1)-3) yields that $[0, \omega]$ is a nonoscillation interval. The conditions on the integral of the function $a(t)$ and reference to the substitution (6.1) completes the proof. \square

7 Floquet Theory and Stability

Consider now the equation

$$\mathcal{L}x(t) \equiv \ddot{x}(t) + a(t)\dot{x}(t) + b(t)x(t) = 0, \quad t \in [0, \infty), \quad (7.1)$$

with ω -periodic coefficients $a(t + \omega) = a(t)$, $b(t + \omega) = b(t)$. For this equation there exist solutions satisfying the condition

$$x(t + \omega) = \lambda x(t). \quad (7.2)$$

The foundations and applications of the Floquet theory were presented in the book by Yakubovich and Starzhinskiĭ [39]. Using the Floquet theory for ordinary differential equations write the equation for λ :

$$\lambda^2 - (x_1(\omega) + x_2'(\omega))\lambda + W(\omega) = 0, \quad (7.3)$$

where x_1 and x_2 are two solutions of the equation (7.1) such that $x_1(0) = 1$, $x_1'(0) = 0$, $x_2(0) = 0$, $x_2'(0) = 1$. Denote by

$$W(t) = \det \begin{pmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{pmatrix}$$

the Wronskian of the fundamental system of (7.1). Obviously $W(0) = 1$.

If λ_1 is a real root of equation (7.2), then the corresponding solution of equation (7.1) has the representation

$$y(t) = g(t) \exp\left(\frac{\ln |\lambda_1|}{\omega} t\right), \quad (7.4)$$

where g is ω -periodic if $\lambda_1 > 0$ and is 2ω -periodic function if $\lambda_1 < 0$. If equation (7.3) has two complex roots $\lambda_1 = |\lambda_1| \exp(i\theta)$ and $\lambda_2 = |\lambda_1| \exp(-i\theta)$, then the corresponding solutions of equation (7.1) have the form

$$y_1(t) = \left(g_1(t) \cos \frac{\theta t}{\omega} - g_2(t) \sin \frac{\theta t}{\omega}\right) \exp\left(\frac{\ln |\lambda_1|}{\omega} t\right), \quad (7.5)$$

$$y_2(t) = \left(g_2(t) \cos \frac{\theta t}{\omega} + g_1(t) \sin \frac{\theta t}{\omega}\right) \exp\left(\frac{\ln |\lambda_1|}{\omega} t\right), \quad (7.6)$$

where g_1 and g_2 are ω -periodic functions.

Theorem 11 *Assume that equation (7.1) is oscillatory and the distance between zeros of its solutions is different from 2ω . Then the following statements are valid.*

a) *Equation (7.1) is exponentially stable if*

$$\int_0^\omega a(t) dt > 0. \quad (7.7)$$

b) *The fundamental solutions of equation (7.1) are of the form (7.4), where $|\lambda_1| > 1$ if*

$$\int_0^\omega a(t) dt < 0. \quad (7.8)$$

c) *If*

$$\int_0^\omega a(t) dt = 0 \quad (7.9)$$

then the fundamental solutions of equation (7.1) are bounded.

Proof. It follows from the classical formula of Ostrogradskii that condition (7.7) implies the inequality $W(\omega) < 1$, condition (7.8) implies the inequality $W(\omega) > 1$, and condition (7.9) implies that $W(\omega) = 1$. The condition that the distance between zeros of solutions of (7.1) is different from 2ω excludes the existence of real roots of equation (7.3). In this case the inequality $W(\omega) < 1$ implies that $|\lambda_1| < 1$, the equality $W(\omega) = 1$ implies that $|\lambda_1| = 1$, and the inequality $W(\omega) > 1$ implies that $|\lambda_1| > 1$. Now the representation of solutions (7.5),(7.6) completes the proof. \square

Remark. The condition that the distance between zeros are different from 2ω is essential as the following example demonstrates.

Example 4. Consider the equation

$$\mathcal{L}x(t) \equiv x''(t) + \frac{2 \sin^2 t + \cos t \sin t}{1 + \cos t \sin t} x'(t) + \frac{\sin^2 t - \cos t \sin t}{1 + \cos t \sin t} x(t) = 0, \quad t \in [0, \infty). \quad (7.10)$$

Inequality (7.7) for the coefficient $a(t) = \frac{2\sin^2 t + \cos t \sin t}{1 + \cos t \sin t}$ is fulfilled with $\omega = \pi$, but this equation is not exponentially stable: its fundamental system is $x_1 = e^{-t} \cos t$ and $x_2 = \sin t$.

Using the substitution (6.1), we again obtain the equation

$$z''(t) + p(t)z(t) = 0, \quad t \in [0, \infty), \quad (7.11)$$

where

$$p(t) = b(t) - \frac{a^2(t)}{4} - \frac{a'(t)}{2}, \quad (7.12)$$

Evidently zeros of the solution x of equation (7.1) and the corresponding solution z of the equation (6.3) coincide. Let us denote

$$P = \operatorname{ess\,inf}_{t \in [0, \omega]} p(t), \quad Q = \operatorname{ess\,sup}_{t \in [0, \omega]} p(t). \quad (7.13)$$

Estimating distances between two adjacent zeros (i.e. nonoscillation intervals) from below and from above we get the following result.

Theorem 12 *Suppose $P > 0$, there exists a positive integer k such that $\frac{k-1}{k} < \sqrt{\frac{P}{Q}}$ and*

$$\omega \in \left(0, \frac{\pi}{2\sqrt{Q}}\right] \cup \left(\frac{1}{2}\frac{\pi}{\sqrt{P}}, \frac{\pi}{\sqrt{Q}}\right) \cup \dots \cup \left(\frac{k-1}{2}\frac{\pi}{\sqrt{P}}, \frac{k\pi}{2\sqrt{Q}}\right). \quad (7.14)$$

Then equation (7.1) is oscillatory and distance between zeros of its solutions is different from 2ω .

Proof. If equation (7.3) has real roots, then there exist such zeros t_0, t_1 of a solution $x(t)$ that the distance between t_0 and t_1 equals ω or 2ω . We will reject this possibility, since the distance between zeros of $g(t)$ in (7.4) cannot be 2ω .

Assume that $x(t_0) = 0$. We use the functions $v = \sin \sqrt{Q}(t - t_0)$ in the first assertion of Theorem A to get that the spectral radius of the operator $K_{t_0, t_0 + \frac{\pi}{2\sqrt{Q}}} : C_{[t_0, t_0 + \frac{\pi}{2\sqrt{Q}}]} \rightarrow C_{[t_0, t_0 + \frac{\pi}{2\sqrt{Q}}]}$ defined by the equality

$$K_{t_0, t_0 + \frac{\pi}{2\sqrt{Q}}} x(t) = - \int_{t_0}^{t_0 + \frac{\pi}{2\sqrt{Q}}} G_{t_0, t_0 + \frac{\pi}{2\sqrt{Q}}}(t, s) p(s) x(s) ds. \quad (7.15)$$

where $G_{t_0, t_0 + \frac{\pi}{2\sqrt{Q}}}(t, s)$ is the Green's function of the problem

$$x''(t) + a(t)x'(t) + b(t)x(t) = f(t), \quad t \in [0, \omega], \quad x(t_0) = 0, \quad x\left(t_0 + \frac{\pi}{2\sqrt{Q}}\right) = 0, \quad (7.16)$$

is less than one.

Applying Theorem 5.4 in [25], p. 81, we obtain that the spectral radius of the operator $K_{t_0, t_0 + \frac{\pi}{2\sqrt{P}}} : C_{[t_0, t_0 + \frac{\pi}{2\sqrt{P}}]} \rightarrow C_{[t_0, t_0 + \frac{\pi}{2\sqrt{P}}]}$, defined by the equality

$$K_{t_0, t_0 + \frac{\pi}{2\sqrt{P}}} x(t) = - \int_{t_0}^{t_0 + \frac{\pi}{2\sqrt{P}}} G_{t_0, t_0 + \frac{\pi}{2\sqrt{P}}}(t, s) p(s) x(s) ds. \quad (7.17)$$

where $G_{t_0, t_0 + \frac{\pi}{2\sqrt{P}}}(t, s)$ is the Green's function of the problem

$$x''(t) + a(t)x'(t) + b(t)x(t) = f(t), \quad t \in [0, \omega], \quad x(t_0) = 0, \quad x\left(t_0 + \frac{\pi}{2\sqrt{P}}\right) = 0, \quad (7.18)$$

is greater or equal to one. Moving the point t_0 we obtain that there are no zeros in the zones defined by (7.14). \square

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